2.7/2.9 Take 2

I sorta covered the topics of 2.9 as 2.7, so now let's cover some examples of turning symbolic sentences into English (2.7).

Ex: Let $P_n(\mathbb{R})$ be the set of polynomials of degree n with coefficients in $\mathbb{R}$.

ⓐ $\forall n \in \mathbb{N}, \forall f \in P_n(\mathbb{R}), \exists r \in \mathbb{R}, f(r) = 0$

ⓑ $\forall n \in \mathbb{N}, \forall f \in P_n(\mathbb{R}), \exists r \in \mathbb{C}, f(r) = 0$

ⓐ Every polynomial with real coefficients and degree at least one has a real root.
This is false, e.g., $f(x) = x^2 + 1$.

ⓑ Every polynomial over $\mathbb{R}$ of degree at least one has a complex root.
The Fundamental Theorem of Algebra

Ex: ⓐ $\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{N}, nx > y$

ⓑ $\forall x, y \in \mathbb{R}, x < y, \exists p \in \mathbb{Q}, x < p < y$

ⓐ Given real numbers x, y with $x > 0$, there is a positive integer n such that $nx > y$.

ⓑ Between any two real numbers, there is a rational number.

Ex: Let $E=\{4,6,8,10,12,\cdots\}$ and let $P=\{p \mid p \text{ is prime}\}$

Rewrite in symbols: (Goldbach's Conjecture)

"Every even integer greater than 2 is the sum of two primes."

$$(n\in E)\Rightarrow(\exists p,q\in P,\ n=p+q)$$

(or)

$$\forall n\in E, \exists p,q\in P,\ n=p+q$$

2.10: Negating Statements

There are some important rules when negating:

DeMorgan's Laws $\quad \sim(P\vee Q)=(\sim P)\wedge(\sim Q)$

$\sim(P\wedge Q)=(\sim P)\vee(\sim Q)$

Negation of $\Rightarrow$ $\quad \sim(P\Rightarrow Q)=P\wedge(\sim Q)$

Ex: ⓐ The hat is red or blue.

ⓑ The book is large and heavy.

ⓒ If x is rational, then $\sqrt{x}$ is algebraic.

Sol:

ⓐ The hat is not red and the hat is not blue.

ⓑ The book is not large or the book is not heavy

ⓒ x is rational and $\sqrt{x}$ is not algebraic

Now, what about negating statements with $\forall$ and $\exists$?

$$\sim(\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x)$$
$$\sim(\exists x \in S, P(x)) \equiv \forall x \in S, \sim P(x)$$

Ex: (a) $\forall x \in \mathbb{Q}, x^2 \in \mathbb{Q}$

(b) $\exists x \in \mathbb{C}, x^2+1=0$

(c) $\forall y \in Y, \exists x \in X, f(x)=y$

(d) $(f(x)=f(y)) \Rightarrow (x=y)$

(e) The square of an odd number is odd.

(f) There is a rational number whose square is π.

Sol: (a) $\exists x \in \mathbb{Q}, x^2 \notin \mathbb{Q}$

(b) $\forall x \in \mathbb{C}, x^2+1 \neq 0$

(c) $\exists y \in Y, \forall x \in X, f(x) \neq y$

(d) $(f(x)=f(y)) \wedge (x \neq y)$

(e) There is an odd number whose square is not odd

(f) There is no rational number whose square is π.